JOURNAL OF APPROXIMATION THEORY 40, 202–215 (1984)

Lipschitz Conditions, Strong Uniqueness, and Almost Chebyshev Subspaces of $C(X)$

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Communicated by Oved Shisha

Received April 29, 1982

For a finite dimensional subspace M of $C(X)$, X a compact metric space, it is well known that the (set valued) metric projection P_M is (Hausdorff) continuous at any $f \in C(X)$ having a unique best approximation from M and is point Lipschitz continuous at any $f \in C(X)$ having a strongly unique best approximation from M. The converses of these classical results are studied. It is shown that if f has a unique best approximation and P_M is point Lipschitzian at f, then f has a strongly unique best approximation. If M is an almost Chebyshev subspace of $C(X)$, then the converses of both statements above are shown to hold. Using a theorem of Garkavi, the validity of these converses actually characterizes the almost Chebyshev subspaces of $C(X)$.

1. INTRODUCTION

Let $C(X)$ denote the space of continuous, real-valued functions on the compact metric space X endowed with the uniform norm, and let M be a finite dimensional subspace of $C(X)$. For $f \in C(X)$, $P_M(f)$ shall denote the set of best uniform approximations to f from M. This paper investigates the relationship between strong uniqueness (resp. uniqueness) of best approximations from M and point Lipschitz continuity (resp. continuity) of the metric projection $P_M: C(X) \to 2^M$.

We say that $m^* \in M$ is a *unique best approximation* to $f \in C(X)$ from M if $P_M(f) = \{m^*\}$ or, equivalently,

$$\|f - m\| > \|f - m^*\|$$

0021-9045/84 $3.00

for all $m \in M \backslash \{m^*\}$. We say that m^* is a *strongly unique best approximation* to f if there is a constant $\gamma = \gamma(f) > 0$ such that

$$\|f - m\| \geqslant \|f - m^*\| + \gamma \|m - m^*\|$$

for all $m \in M$. We consider continuity with respect to the Hausdorff metric

$$h(U, V) = \max \{\sup_{u \in U} \inf_{v \in V} \|u - v\|, \sup_{v \in V} \inf_{u \in U} \|u - v\|\}, \tag{1.1}$$

where U and V are closed, nonempty subsets of $C(X)$. The metric projection P_M is said to be (Hausdorff) *continuous* at $f \in C(X)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$h(P_M(f), P_M(g)) < \varepsilon \tag{1.2}$$

for all $g \in C(X)$ with $\|f - g\| < \delta$. We say that P_M is *point Lipschitz continuous* (or *point Lipschitzian*) at f there is a constant $\lambda = \lambda(f) > 0$ such that

$$h(P_M(f), P_M(g)) \leqslant \lambda \|f - g\| \tag{1.3}$$

for all $g \in C(X)$.

The following theorem gives known relationships between these concepts. The first statement is probably folklore and the second is essentially due to Cheney [2, p. 82].

THEOREM 1. (a) *If $f \in C(X)$ has a unique best approximation from M, then P_M is continuous at f.* (b) *If f has a strongly unique best approximation from M, then P_M is point Lipschitz continuous at f.*

In this paper we enquire as to whether the converses of the statements in Theorem 1 are valid. In Section 2, Theorem 2, we show that if f has a unique best approximation from M and P_M is point Lipschitzian at f, then f has a strongly unique best approximation. To obtain precise converses, we require an additional condition on M. We say that M is a *Chebyshev subspace* of $C(X)$ if every function in $C(X)$ has a unique best approximation from M and that M is an *almost Chebyshev subspace* of $C(X)$ if except for a set of first category in $C(X)$ every function has a unique best approximation from M (see Garkavi [3, 4]). In Theorem 3, we show that if M is almost Chebyshev, then the functions in $C(X)$ at which P_M is continuous are precisely those that have unique best approximations and the functions at which P_M is point Lipschitz continuous are precisely those that have strongly unique best approximations.

In Section 3, Theorem 4, we show that in fact the converse of either statement in Theorem 1 constitutes a complete characterization of finite dimensional almost Chebyshev subspaces of $C(X)$. The "point Lipschitz continuity implies strong uniqueness" characterization of almost Chebyshev subspaces has an interesting counterpart for Chebyshev subspaces. McLaughlin and Somers [6] have shown that "uniqueness implies strong uniqueness" completely characterizes Chebyshev subspaces of $C(X)$.

The results of this paper are surprising when one considers the same questions in Hilbert spaces or in the L_p spaces with $2 < p < \infty$. It is well known that metric projections onto closed subspaces of a Hilbert space satisfy Lipschitz conditions, and Holmes and Kripke [5] have shown that metric projections onto finite dimensional subspaces of L_p $(2 < p < \infty)$ are pointwise Lipschitzian. However, these spaces are smooth, and Wulbert [10] has shown that no nontrivial subspace of a smooth space admits strongly unique best approximations to points not in the subspace. Thus Theorem 2 fails in both spaces. In both cases, finite dimensional subspaces are Chebyshev subspaces and hence the characterization of almost Chebyshev subspaces in Section 3 does not extend to these spaces.

The Hausdorff metric (1.1) is used in (1.3) to obtain greater generality than other authors in the definition of point Lipschitz continuity of a set valued metric projection. Specifically, Nürnberger [7] defines P_M to be point Lipschitzian at f if f has a unique best approximation m_f from M and there is a constant $\lambda = \lambda(f) > 0$ such that $\|m_f - m_g\| \leqslant \lambda \|f - g\|$ for all $g \in C(X)$ and $m_g \in P_M(g)$. If f has a unique best approximation from M, then (1.3) and Nürnberger's definition coincide. Theorem 2 then asserts that point Lipschitz continuity of P_M at f in the sense of Nürnberger implies that f has a strongly unique best approximation. No generality is lost in using the Hausdorff metric in (1.2) since the concepts of continuity, Hausdorff continuity, and Kuratowski continuity coincide when M is finite dimensional (see Singer [8]).

2. Point Lipschitz Continuity and Strong Uniqueness

In this section, we establish converses of the statements in Theorem 1. We first introduce some notations and cite the main theorems that will be used.

Let $\{m_1, \ldots, m_n\}$ be a basis for M, and for $x \in X$, let $\hat{x} = (m_1(x), \ldots, m_n(x)) \in R^n$ and $\theta = (0, \ldots, 0) \in R^n$. For $A \subseteq R^n$, co A, int co A, and bdy co A shall denote the convex hull, the interior of the convex hull, and the boundary of the convex hull of A, respectively. For $f \in C(X)$, the extreme set of f is defined to be

$$E(f) = \{x \in X : |f(x)| = \|f\|\}.$$

It is well known that $m^* \in M$ is a best approximation to $f \in C(X)$ from M if and only if

$$\theta \in \text{co}\{(f-m^*)(x)\,\hat{x} : x \in E(f-m^*)\}.$$

(See Cheney [2, p. 73].) Our main tool is a similar characterization of strong uniqueness due to Bartelt and MacLaughlin [1]. That is, $m^* \in M$ is a strongly unique best approximation to f from M if and only if

$$\theta \in \text{int co}\{(f-m^*)(x)\,\hat{x}\} : x \in E(f-m^*)\}.$$

THEOREM 2. *Let $f \in C(X)$. If f has a unique best approximation from M and P_M is point Lipschitz continuous at f, then the best approximation to f is strongly unique.*

Proof. Without loss of generality, we suppose that 0 is the unique best approximation to f and that $\|f\| = 1$. We assume that 0 is not a strongly unique best approximation to f and show that P_M cannot be point Lipschitzian at f. By the theorems cited above, we have that

$$\theta \in \text{co}\{f(x)\,\hat{x} : x \in E(f)\} \cap \text{bdy co}\{f(x)\,\hat{x} : x \in E(f)\}. \tag{2.1}$$

By a corollary to the supporting hyperplane theorem (see Stoer and Witzgall [9, p. 103]), there is a nonzero linear functional L on R^n, say

$$L(\xi_1, ..., \xi_n) = \sum_{j=1}^{n} \alpha_j \xi_j,$$

such that

$$L(\theta) = 0 \geqslant L(f(x)\,\hat{x}) \tag{2.2}$$

for all $x \in E(f)$. Define the function $m \in M$ by

$$m(x) = \sum_{j=1}^{n} \alpha_j m_j(x). \tag{2.3}$$

It follows from (2.2) that

$$f(x)\, m(x) \leq 0 \qquad \text{for} \quad x \in E(f). \tag{2.4}$$

Now by (2.1), there is a finite set of points $z_1, ..., z_k$ in $E(f)$ such that

$$\theta \in \text{co}\{f(z_i)\,\hat{z}_i : i = 1, ..., k\}, \tag{2.5}$$

and (2.4) and (2.5) imply that

$$m(z_i)=0 \qquad (i=1,\dots,k). \tag{2.6}$$

For $\delta>0$, we construct a function $g_\delta \in C(X)$ and a number $\lambda=\lambda(\delta)>0$ such that $-\lambda m \in P_M(g_\delta)$, $\|f-g_\delta\|=\delta$, and $\lim_{\delta\to 0^+}\lambda(\delta)/\delta=+\infty$.

Let $Z=\{x\in E(f): m(x)=0\}$, $Z^+=\{x\in Z: f(x)=1\}$, and $Z^-=\{x\in Z: f(x)=-1\}$. Evidently, $Z=Z^+\cup Z^-$ and each of these sets is closed in X. Since X is normal, there is a closed neighborhood $\mathcal{N}^+$ of Z^+ such that $f(x)>\frac{1}{2}$ for $x\in\mathcal{N}^+$ and there is a closed neighborhood $\mathcal{N}^-$ of Z^- such that $f(x)<-\frac{1}{2}$ for $x\in\mathcal{N}^-$. Let $\mathcal{N}=\operatorname{int}\mathcal{N}^+\cup\operatorname{int}\mathcal{N}^-$. Then $\mathcal{N}$ is an open neighborhood of Z. We require

LEMMA 1. *For every* $\delta>0$, *there is a number* $\lambda=\lambda(\delta)>0$ *such that* $\operatorname{sgn} f(x)(f(x)+\lambda m(x))\leqq 1+\delta$ *for* $x\in\mathcal{N}$, $\lim_{\delta\to 0^+}\lambda(\delta)=0$, *and* $\lim_{\delta\to 0^+}\lambda(\delta)/\delta=+\infty$.

Proof of Lemma 1. We first show that for each $\delta>0$, we may select $\lambda^+(\delta)>0$ such that $\operatorname{sgn} f(x)(f(x)+\sigma m(x))\leqq 1+\delta$ for $x\in\mathcal{N}^+$ and $0<\sigma\leqslant\lambda^+(\delta)$, $\lim_{\delta\to 0^+}\lambda^+(\delta)=0$, and $\lim_{\delta\to 0^+}\lambda^+(\delta)/\delta=+\infty$.

Let $T=\{x\in\mathcal{N}^+: m(x)\leqq 0\}$ and $S=\mathcal{N}^+\setminus T$. If $S=\varnothing$, then $m(x)\leqq 0$ for all $x\in\mathcal{N}^+$. In this case, let $\lambda^+(\delta)=\sqrt{\delta}$, and $\operatorname{sgn} f(x)(f(x)+\sigma m(x))=f(x)+\sigma m(x)\leqq 1$ for $x\in\mathcal{N}^+$ and $0<\sigma\leqslant\lambda^+(\delta)$. The other two conditions are clear. Suppose $S\neq\varnothing$. Let

$$\lambda^+(\delta)=\inf_{t\in S}\frac{\delta+1-f(t)}{m(t)}. \tag{2.7}$$

We first verify that the infimum in (2.7) is attained. Since $S\neq\varnothing$, $\lambda^+(\delta)<\infty$. Select a sequence $\{t_j\}$ in S such that

$$\frac{\delta+1-f(t_j)}{m(t_j)}\to\lambda^+(\delta).$$

Since X is compact, we may assume that $t_j\to x_\delta\in X$. Since $\mathcal{N}^+$ is closed, $x_\delta\in\mathcal{N}^+$. If $x_\delta\in T$, then $m(x_\delta)\leqq 0$ and the continuity of m would imply that $m(x_\delta)=0$. Thus $m(t_j)\to 0$ and hence

$$\frac{\delta+1-f(t_j)}{m(t_j)}\geqslant\frac{\delta}{m(t_j)}\to+\infty$$

which contradicts the fact that $\lambda^+(\delta) < \infty$. Thus $x_\delta \in S$, $m(x_\delta) > 0$, and the continuity of f and m imply that

$$\lambda^+(\delta) = \frac{\delta + 1 - f(x_\delta)}{m(x_\delta)}. \tag{2.8}$$

Moreover, it now follows from (2.8) that $\lambda^+(\delta) > 0$.

We next show that $\lim_{\delta \to 0^+} \lambda^+(\delta)/\delta = +\infty$. It suffices to show that for every sequence $\{\delta_j\}$ of positive numbers with $\delta_j \to 0$ there is a subsequence $\{\delta_{j_\nu}\}$ such that $\lambda^+(\delta_{j_\nu})/\delta_{j_\nu} \to +\infty$. In the remainder of this paragraph we suppress the subscripts on the symbol δ. Suppose $\delta \to 0$. We extract a subsequence so that $x_\delta \to x \in \mathcal{N}^+$. Either $x \in T$ or $x \in S$. If $x \in T$, then as above $m(x) = 0$, $m(x_\delta) \to 0$, and

$$\frac{\lambda^+(\delta)}{\delta} = \left(1 + \frac{1 - f(x_\delta)}{\delta}\right) \Big/ m(x_\delta) \geqslant \frac{1}{m(x_\delta)} \to +\infty.$$

If $x \in S$, then $m(x) > 0$. If $f(x) = 1$, then $x \in E(f)$ and $f(x)\, m(x) > 0$ contrary to (2.4). Thus $f(x) < 1$ and

$$\frac{\lambda^+(\delta)}{\delta} = \left(1 + \frac{1 - f(x_\delta)}{\delta}\right) \Big/ m(x_\delta) \geqslant \frac{1 - f(x_\delta)}{\delta m(x_\delta)} \to +\infty.$$

Hence, $\lim_{\delta \to 0^+} \lambda^+(\delta)/\delta = +\infty$.

Now by (2.7) and the argument above (2.7), $\operatorname{sgn} f(x)(f(x) + \sigma m(x)) \leqq 1 + \delta$ for $x \in \mathcal{N}^+$ and $0 < \sigma \leqslant \lambda^+(\delta)$. Finally, if $\limsup_{\delta \to 0^+} \lambda^+(\delta) > 0$, we may replace $\lambda^+(\delta)$ by $\min(\sqrt{\delta}, \lambda^+(\delta))$ and the result holds.

Replacing f and m by $-f$ and $-m$ in the argument above, we see that for $\delta > 0$, there is a number $\lambda^-(\delta) > 0$ such that $\operatorname{sgn} f(x)(f(x) + \sigma m(x)) \leqslant 1 + \delta$ for $x \in \mathcal{N}^-$ and $0 < \sigma \leqslant \lambda^-(\delta)$, $\lim_{\delta \to 0^+} \lambda^-(\delta) = 0$, and $\lim_{\delta \to 0^+} \lambda^-(\delta)/\delta = +\infty$. The lemma is proven by letting $\lambda(\delta) = \min(\lambda^+(\delta), \lambda^-(\delta))$.

Returning to the proof of Theorem 2, we now construct the function g_δ. Since $z_i \in Z$ $(i = 1,..., k)$ and $\mathcal{N}$ is an open neighborhood of Z, (2.6) implies that there is an open neighborhood G_δ of $\{z_1,..., z_k\}$ such that

$$\lambda\, |m(x)| < \delta/2 \qquad \text{for} \quad x \in G_\delta \tag{2.9}$$

and

$$G_\delta \subseteq \mathcal{N}. \tag{2.10}$$

Now define a real-valued function φ_δ on $\{z_1,..., z_k\} \cup (X \backslash G_\delta)$ by

$$\varphi_\delta(z_i) = \delta \qquad (i = 1,..., k) \tag{2.11}$$

and

$$\varphi_\delta(x) = 0 \qquad \text{for} \quad x \in X \backslash G_\delta. \tag{2.12}$$

Observe that φ_δ is continuous and satisfies the inequality

$$0 \leqslant \varphi_\delta(x) \leqslant |\delta - \lambda\, |m(x)|| \tag{2.13}$$

on the closed set $\{z_1, ..., z_k\} \cup (X \backslash G_\delta)$. A mild variation of the Tietze extension theorem allows us to extend φ_δ continuously to all of X so that (2.13) holds for all $x \in X$. Finally, we define $g_\delta \in C(X)$ by

$$g_\delta(x) = f(x) + f(x)\, \varphi_\delta(x). \tag{2.14}$$

We show that $\|f - g_\delta\| = \delta$. For $x \in X \backslash G_\delta$, (2.12) and (2.14) imply that $g_\delta(x) = f(x)$. For $x \in G_\delta$, (2.9), (2.13), and (2.14) yield

$$|f(x) - g_\delta(x)| \leqslant |\varphi_\delta(x)| \leqslant |\delta - \lambda\, |m(x)|| \leqslant \delta.$$

Now $f(z_1) - g_\delta(z_1) = -f(z_1)\, \varphi_\delta(z_1) = \pm\delta$ by (2.11) and hence $\|f - g_\delta\| = \delta$.

Next we verify that $-\lambda m \in P_M(g_\delta)$ for δ sufficiently small where λ is given by Lemma 1. The set $E(f) \backslash \mathscr{N}$ is closed, and (2.4) and the fact that $Z \subseteq \mathscr{N}$ imply that $f(x)\, m(x) < 0$ for $x \in E(f) \backslash \mathscr{N}$. Thus there is an open neighborhood $\mathscr{F}$ of $E(f) \backslash \mathscr{N}$ such that $|f(x)| > \frac{1}{2}$ and $f(x)\, m(x) < 0$ for $x \in \mathscr{F}$. Since $\mathscr{N} \cup \mathscr{F}$ is open and covers $E(f)$,

$$\mu := \sup_{x \in X \backslash (\mathscr{N} \cup \mathscr{F})} |f(x)| < 1.$$

By Lemma 1, $\lim_{\delta \to 0^+} \lambda = 0$, and so there is a $\delta_0 > 0$ such that $\lambda\, \|m\| < \min(1/2, 1 - \mu)$ for $0 < \delta \leqq \delta_0$. Suppose $0 < \delta \leqslant \delta_0$. We first show that $\|g_\delta + \lambda m\| = 1 + \delta$. For $x \in X \backslash (\mathscr{N} \cup \mathscr{F})$, (2.12) yields

$$\begin{aligned} |g_\delta(x) + \lambda m(x)| &= |f(x) + \lambda m(x)| \leqq |f(x)| + \lambda\, |m(x)| \\ &\leqslant \mu + 1 - \mu = 1. \end{aligned}$$

For $x \in \mathscr{F} \backslash \mathscr{N}$, $\lambda\, |m(x)| < \frac{1}{2} < |f(x)|, f(x)\, m(x) < 0$, and (2.12) imply that

$$|g_\delta(x) + \lambda m(x)| = |f(x) + \lambda m(x)| = |f(x)| - \lambda\, |m(x)| \leqslant 1.$$

For $x \in \mathscr{N} \backslash G_\delta$, $\lambda\, |m(x)| < \frac{1}{2} < |f(x)|$, (2.12), and Lemma 1 ensure that

$$|g_\delta(x) + \lambda m(x)| = |f(x) + \lambda m(x)| = \operatorname{sgn} f(x) (f(x) + \lambda m(x)) \leqslant 1 + \delta.$$

For $x \in G_\delta$, (2.9) and (2.13) yield

$$| g_\delta(x) + \lambda m(x)| \leqslant |f(x)| + |\varphi_\delta(x)| + \lambda\, |m(x)|$$
$$\leqslant |f(x)| + \delta - \lambda\, |m(x)| + \lambda\, |m(x)| \leqq 1 + \delta.$$

Finally, for $i = 1,..., k$, (2.6) and (2.11) imply that

$$g_\delta(z_i) + \lambda m(z_i) = (1 + \delta) f(z_i) = \pm(1 + \delta).$$

Thus $\| g_\delta + \lambda m\| = 1 + \delta$ and $\{z_1,..., z_k\} \subseteq E(g_\delta + \lambda m)$. By (2.5),

$$\theta \in \text{co}\{(1 + \delta) f(z_i)\, \hat{z}_i : i = 1,...,k\} \subseteq \text{co}\{(g_\delta + \lambda m)(x)\, \hat{x} : x \in E(g_\delta + \lambda m)\}.$$

By the theorem on p. 73 in Cheney [2], $-\lambda m \in P_M(g_\delta)$.

Since $P_M(f) = \{0\}$,

$$\frac{h(P_M(f), P_M(g_\delta))}{\|f - g_\delta\|} \geqslant \frac{\|-\lambda m - 0\|}{\delta} = \frac{\lambda\, \|m\|}{\delta} \to +\infty$$

as $\delta \to 0$ by Lemm 1. Thus P_M is not point Lipschitzian at f. The proof of Theorem 2 is now complete.

We now turn to the case in which M is an almost Chebyshev subspace of $C(X)$. The importance of this condition is that if M is almost Chebyshev, then the set of functions that have unique best approximations from M is dense in $C(X)$. It is of interest to note that Garkavi [3] has shown that in $C(X)$ (in fact, in all separable spaces) the almost Chebyshev property for reflexive subspaces is equivalent to the set of functions having unique best approximations being dense in $C(X)$.

THEOREM 3. *Let M be an almost Chebyshev subspace of $C(X)$.* (a) *If P_M is (Hausdorff) continuous at $f \in C(X)$, then f has a unique best approximation from M.* (b) *If P_M is point Lipschitz continuous at $f \in C(X)$, then f has a strongly unique best approximation from M.*

Proof. We need only prove (a) for (b) follows from (a) and Theorem 2. Suppose f does not have a unique best approximation from M. Select distinct $u, v \in P_M(f)$. Since M is almost Chebyshev, there is a sequence $\{g_k\}$ in $C(X)$ such hat $\|f - g_k\| \to 0$ and $P_M(g_k) = \{m_k\}$. That is, each g_k has a unique best approximation from M. Then

$$0 < \tfrac{1}{2}\, \|u - v\| \leqslant \tfrac{1}{2}\{\|u - m_k\| + \|m_k - v\|\}$$
$$\leqslant \max\{\|u - m_k\|, \|v - m_k\|\} \leqq \sup_{w \in P_M(f)} \|w - m_k\|$$
$$\leqq h(P_M(f), P_M(g_k)).$$

So $h(P_M(f), P_M(g_k)) \not\to 0$ and P_M is not continuous at f.

3. A CHARACTERIZATION OF ALMOST CHEBYSHEV SUBSPACES OF $C(X)$

The aim of this section is to investigate the role of the almost Chebyshev condition on M in Theorem 3. We shall see that properties (a) and (b) in Theorem 3 completely characterize finite dimensional almost Chebyshev subspaces of $C(X)$.

Our main tool in this section is a Haar-like characterization of almost Chebyshev subspaces of $C(X)$ due to Garkavi [4]. If $G \subseteq X$, let

$$N_n(G) = \operatorname{card}(G), \quad \text{if} \quad \operatorname{card}(G) \leqslant n,$$
$$= n, \quad \text{otherwise.}$$

Garkavi showed that an n-dimensional subspace M of $C(X)$ is almost Chebyshev if and only if for every open subset G of X, at most $n - N_n(G)$ linearly independent functions in M vanish identically on G. If X has no isolated points, this condition reduces to the property that no nonzero element of M can vanish identically on a nonempty open subset of X.

We also require the following lemma which asserts that local point Lipschitz continuity is equivalent to global point Lipschitz continuity for metric projections.

LEMMA 2. *The metric projection P_M is point Lipschitz continuous at $f \in C(X)$ if and only if there exist constants $\lambda > 0$ and $\varepsilon > 0$ such that $h(P_M(f), P_M(g)) \leqslant \lambda \|f-g\|$ for all $g \in C(X)$ with $\|f-g\| \leqq \varepsilon$.*

Proof. The "only if" part is clear. Suppose such constants λ and ε exist. Let $g \in C(X)$ with $\|f-g\| > \varepsilon$. Let $u \in P_M(f)$ and $v \in P_M(g)$. If $\|f-g\| \geqslant \|f\|$, then

$$\frac{\|u-v\|}{\|f-g\|} \leqslant \frac{2(\|f\| + \|g\|)}{\|f-g\|} \leqslant \frac{2(2\|f\| + \|f-g\|)}{\|f-g\|} \leqslant 6.$$

If $\|f-g\| < \|f\|$, then

$$\frac{\|u-v\|}{\|f-g\|} \leqslant \frac{1}{\varepsilon} \|u-v\| \leqslant \frac{2}{\varepsilon} (\|f\| + \|g\|)$$
$$\leqslant \frac{2}{\varepsilon} (2\|f\| + \|f-g\|) \leqslant 6\|f\|/\varepsilon.$$

Hence, $h(P_M(f), P_M(g)) \leqslant \max(6, 6\|f\|/\varepsilon)\|f-g\|$, and thus P_M is point Lipschitzian at f with Lipschitz constant $\max(\lambda, 6, 6\|f\|/\varepsilon)$.

THEOREM 4. *Let M be a finite dimensional subspace of $C(X)$. The following are equivalent*:

(1) *M is almost Chebyshev.*

(2) *If P_M is continuous at f, then f has a unique best approximation from M.*

(3) *If P_M is point Lipschitzian at f, then f has a strongly unique best approximation from M.*

Proof. Theorems 3 and 2 yield (1) implies (2) and (2) implies (3), respectively. We prove (3) implies (1).

We suppose that M is not almost Chebyshev and show that (3) fails. The initial reductions in this proof are identical to the first steps in the proof of necessity in Theorem 1 in Garkavi [4, pp. 181–183], and we refer the reader to his paper for the details. Let $\{m_1,..., m_n\}$ be a basis for M. If M is not almost Chebyshev, then the Haar-like condition cited above fails. Garkavi's reduction then yields one of the following

(A) On some open subset G of X containing at least $l+1$ points, the number of linearly independent functions in $\{m_1,..., m_n\}$ over G is l, where $0<l<n$. These functions remain linearly independent over any open subset G' of G containing at least l points.

(B) On some nonempty open subset G of X, all the functions $m_1,..., m_n$ vanish identicaly on G.

We consider case (A). Without loss of generality, $m_1,..., m_l$ are the functions which are linearly independent over G. For $i>l$, we have that m_i is linearly dependent on $m_1,..., m_n$ over G, and thus there exist constants $\alpha_1,..., \alpha_l$ such that $m_i' = m_i - \sum_{j=1}^{l} \alpha_j m_j$ vanishes identically on G. We also have that $\{m_1,..., m_l, m_{l+1}',..., m_n'\}$ is a basis for M. Again, as Garkavi has shown, there exist $l+1$ points $x_0,..., x_l$ in G such that the determinant

$$D_k = \begin{vmatrix} m_1(x_0) & \cdots & m_l(x_0) \\ m_1(x_{k-1}) & \cdots & m_l(x_{k-1}) \\ m_1(x_{k+1}) & \cdots & m_l(x_{k+1}) \\ m_1(x_l) & \cdots & m_l(x_l) \end{vmatrix} \tag{3.1}$$

is nonzero for $k=0,..., l$.

Let $M_1 = \text{sp}\{m_1,..., m_l\}$ and $M_2 = \text{sp}\{m_{l+1}',..., m_n'\}$. Then $M = M_1 \oplus M_2$. Let $B_2 = \{m \in M_2 : \|m\| \leqslant 1\}$ be the unit ball in M_2. By the selection of $m_{l+1}',..., m_n'$, we see that

$$m|_G = 0 \qquad \text{for} \quad m \in M_2. \tag{3.2}$$

The rest of the proof relies heavily on

LEMMA 3. *There exist signs $\sigma_0,\dots,\sigma_l \in \{-1, 1\}$ and a positive constant K such that if $m \in M_1$, $\delta > 0$, and*

$$\sigma_i m(x_i) \leqslant \delta \qquad (i = 0,\dots, l),$$

then $\|m\| \leqq K\delta$.

Proof of Lemma 3. We select the signs σ_i as follows. Since each of the determinants in (3.1) are nonzero, given any l points $y_1,\dots, y_l$ in the set $\{x_0,\dots, x_l\}$ and any l real numbers $r_1,\dots, r_l$ there is a unique $m \in M_1$, such that $m(y_i) = r_i$ $(i = 1,\dots, l)$. For $i = 1,\dots, l$, let L_i be the unique element of M_1 satisfying

$$\begin{aligned} L_i(x_j) &= 1, && \text{if} \quad j = i, \\ &= 0, && \text{if} \quad j = 1,\dots, l, j \neq i. \end{aligned}$$

Then $L_i(x_0) \neq 0$ for otherwise L_i would have l zeros in $\{x_0,\dots, x_l\}$ and would therefore be identically zero. Let $\sigma_i = \operatorname{sgn} L_i(x_0)$ $(i = 1,\dots, l)$ and $\sigma_0 = -1$. It now suffices to show that

$$\sup\{\|m\| : m \in M_1, \sigma_i m(x_i) \leqslant 1 \ (i = 0,\dots, l)\} < \infty.$$

Suppose that $\{p_\nu\}$ is a sequence in M_1 such that $\sigma_i p_\nu(x_i) \leqslant 1$ $(i = 0,\dots, l)$ and $\|p_\nu\| \to \infty$. By the interpolating property above, $\max_{1 \leqslant i \leqslant l} |m(x_i)|$ is a norm on M_1 and hence is equivalent to the uniform norm on M_1. Thus

$$\max_{1 \leqslant i \leqslant l} |p_\nu(x_i)| \to \infty.$$

Extracting a subsequence if necessary, we may assume that $\sigma_k p_\nu(x_k) \to -\infty$ for some fixed index $k \in \{1,\dots, l\}$. For fixed $i \in \{1,\dots, l\}$ if $\sigma_i p_\nu(x_i) \geqslant 0$, then $\sigma_i p_\nu(x_i) \leqslant 1$ and

$$|p_\nu(x_i) L_i(x_0)| \leqslant |L_i(x_0)|.$$

If $\sigma_i p_\nu(x_i) < 0$, then

$$p_\nu(x_i) L_i(x_0) < 0 < |L_i(x_0)|.$$

Thus

$$p_\nu(x_0) \leqslant p_\nu(x_k) L_k(x_0) + \sum_{\substack{i=1 \\ i \neq k}}^{l} |L_i(x_0)| \to -\infty$$

as $\nu \to \infty$. This contradicts the fact that $\sigma_0 p_k(x_0) \leqslant 1$ and Lemma 3 is now proven.

Returning to the proof of Theorem 4, we select $f \in C(X)$ satisfying

$$f(x_i) = \sigma_i \qquad (i = 0, \ldots, l), \tag{3.3}$$

$$f(x) = 0 \qquad \text{for} \quad x \in X \backslash G, \tag{3.4}$$

and

$$|f(x)| \leqslant 1 \qquad \text{for} \quad x \in X. \tag{3.5}$$

We show that P_M is point Lipschitzian at f but f does not have a strongly unique best approximation from M.

It is easy to see that $P_{M_2}(f) = B_2$ and that $\text{dist}(f, M_2) = 1$. We show that $P_M(f) = B_2$. Let $u + v \in M$ where $u \in M_1$, $v \in M_2$, and $\|f - (u + v)\| = \text{dist}(f, M) \leqslant 1$. For $i = 0, \ldots, l$,

$$\begin{aligned} 1 - \sigma_i u(x_i) &= \sigma_i(f(x_i) - u(x_i)) \\ &= \sigma_i(f(x_i) - (u(x_i) + v(x_i))) \leqslant 1. \end{aligned}$$

Thus $\sigma_i(-u(x_i)) \leqslant 0$ $(i = 0, \ldots, l)$. By Lemma 3, $u \equiv 0$. Thus $\|f - v\| \leqslant 1$ and so $v \in P_{M_2}(f) = B_2$. Hence, $\text{dist}(f, M) = 1$ and $P_M(f) = B_2$. Since $l < n$, B_2 is nonsingleton and f does not have a unique best approximation from M. So f does not have a strongly unique best approximation from M.

We now show that P_M satisfies a point Lipschitz condition at f. Let $g \in C(X)$ and $u + v \in P_M(g)$ where $u \in M_1$ and $v \in M_2$. For $i = 0, \ldots, l$,

$$\begin{aligned} \sigma_i(g(x_i) - u(x_i)) &= \sigma_i(g(x_i) - (u(x_i) + v(x_i))) \\ &\leqslant \text{dist}(g, M). \end{aligned}$$

But $\sigma_i f(x_i) = 1 = \text{dist}(f, M)$ and subtracting yields

$$\sigma_i(-u(x_i)) \leqslant \sigma_i(f(x_i) - g(x_i)) + \text{dist}(g, M) - \text{dist}(f, M) \leqslant 2\|f - g\|.$$

By Lemma 3,

$$\|u\| \leqslant 2K \|f - g\|. \tag{3.6}$$

Now using $|\text{dist}(g, M) - \text{dist}(f, M)| \leqslant \|f - g\|$ again, we have

$$\|g - (u + v)\| = \text{dist}(g, M) \leqslant 1 + \|f - g\|.$$

So by (3.6),

$$\|g - v\| \leqslant \|g - (u + v)\| + \|u\| \leqslant 1 + (2K + 1)\|f - g\|.$$

Using (3.2) and (3.4)

$$\begin{aligned}\|v\| &= \max_{X\backslash G} |v(x)| \\ &\leqslant \max_{X\backslash G} |f(x)-g(x)| + \max_{X\backslash G} |g(x)-v(x)| \\ &\leqslant 1 + (2K+2)\,\|f-g\|.\end{aligned}$$

Thus

$$\operatorname{dist}(v, B_2) \leqslant (2K+2)\,\|f-g\|, \tag{3.7}$$

and by (3.6) and (3.7), we have

$$\sup_{w\in P_M(g)} \inf_{m\in P_M(f)} \|w-m\| \leqslant (4K+2)\,\|f-g\|. \tag{3.8}$$

Now suppose $\|f-g\| \leqslant 1/(4K+4)$ and fix $u+v\in P_M(g)$ where $u\in M_1$ and $v\in M_2$. We show that $u+\alpha m\in P_M(g)$ for any $m\in B_2$ and $|\alpha|\leqslant 1-(2K+2)\,\|f-g\|$. Note that $\operatorname{dist}(g, M)\geqslant 1-\|f-g\|$. Then

$$\begin{aligned}\sup_G |g(x)-(u(x)+\alpha m(x))| &= \sup_G |g(x)-u(x)| \\ &= \sup_G |g(x)-(u(x)+v(x))| \leqslant \operatorname{dist}(g, M)\end{aligned}$$

and using (3.4) and (3.6)

$$\begin{aligned}\max_{X\backslash G} |g(x)-(u(x)+\alpha m(x))| &\leqslant \max_{X\backslash G} |(f(x)-g(x)| + \max_{X\backslash G} |u(x)| + \alpha \\ &\leqslant \|f-g\| + 2K\,\|f-g\| + 1 - (2K+2)\,\|f-g\| \\ &= 1 - \|f-g\| \leqslant \operatorname{dist}(g, M).\end{aligned}$$

The assertion is now established. Now for $m\in P_M(f)=B_2$, $u+\alpha m\in P_M(g)$ where $\alpha = 1-(2K+2)\,\|f-g\|$ and using (3.6) again

$$\begin{aligned}\|(u+\alpha m)-m\| &\leqslant \|u\| + 1 - \alpha \\ &\leqslant 2K\,\|f-g\| + 1 - 1 + (2K+2)\,\|f-g\| \\ &= (4K+2)\,\|f-g\|.\end{aligned}$$

Thus

$$\sup_{m\in P_M(f)} \inf_{w\in P_M(g)} \|w-m\| \leqslant (4K+2)\,\|f-g\|.$$

for $\|f-g\| \leqslant 1/(4K+4)$. Thus P_M satisfies a local point Lipschitz condition at f and by Lemma 2, P_M satisfies a global point Lipschitz condition at f. The proof for case (A) is now complete.

In case (B) we select $x_0 \in G$ and define $f \in C(X)$ so that $f(x_0) = 1$, $f(x) = 0$ for $x \in X \backslash G$, and $|f(x)| \leqslant 1$ for $x \in X$. It is easy to see that $P_M(f) = B$, where B is the unit ball of M. If $g \in C(X)$ and $\|f - g\| \leqslant \frac{1}{3}$, it can easily be shown that $\{m \in M: \|m\| \leqslant 1 - 2\|f - g\|\} \subseteq P_M(g) \subseteq \{m \in M: \|m\| \leqslant 1 + 2\|f - g\|\}$ and hence $h(P_M(f), P_M(g)) \leqslant 2\|f - g\|$. The result follows by Lemma 2. The proof of Theorem 4 is now complete.

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